

Solutions* to the Review Problems for the Final Exam, Math 151

Disclaimer: We believe the answers to be correct, but we assume no responsibility to any damage due to possible errors and/or typos.

1. Find the following limits:

$$a) \lim_{x \rightarrow -1} (x^2 - 2x + 1) \quad b) \lim_{x \rightarrow \infty} \frac{3x^2 - 7}{\sqrt{x^2 + 2}} \quad c) \lim_{x \rightarrow 0} \frac{\sin 3x}{2 \sin 5x}$$

$$d) \lim_{x \rightarrow \infty} \frac{3e^x + 4e^{-x}}{5e^x + 4e^{-x}} \quad e) \lim_{x \rightarrow 8^-} \frac{|x - 8|}{x - 8} .$$

Solution to 1.

(a) Plugging in we get $(-1)^2 - 2(-1) + 1 = 4$.

Answer to 1a) 4

(b) “forgetting about the little ones” we get

$$\lim_{x \rightarrow \infty} \frac{3x^2 - 7}{\sqrt{x^2 + 2}} = \lim_{x \rightarrow \infty} \frac{3x^2}{\sqrt{x^2}} = \lim_{x \rightarrow \infty} \frac{3x^2}{x} = \lim_{x \rightarrow \infty} 3x = \infty .$$

Answer to 1b) ∞

(c) When the argument of sin goes to 0, you can replace $\sin w$ by w , hence

$$\lim_{x \rightarrow 0} \frac{\sin 3x}{2 \sin 5x} = \lim_{x \rightarrow 0} \frac{3x}{2(5x)} = \lim_{x \rightarrow 0} \frac{3}{10} = \frac{3}{10} .$$

Answer to 1c) 3/10

d) It is better to replace e^x by another variable, let's call it y .

$$\lim_{x \rightarrow \infty} \frac{3e^x + 4e^{-x}}{5e^x + 4e^{-x}} = \lim_{y \rightarrow \infty} \frac{3y + 4/y}{5y + 4/y}$$

Now ‘forgetting about the little ones’ this equals

$$\lim_{y \rightarrow \infty} \frac{3y}{5y} = \frac{3}{5} .$$

Answer to 1d) 3/5

e) Recall that $|z| = -z$ for $z < 0$, so $|x - 8| = -(x - 8)$ for $x < 8$. We have

$$\lim_{x \rightarrow 8^-} \frac{|x - 8|}{x - 8} = \lim_{x \rightarrow 8^-} \frac{-(x - 8)}{x - 8} = \lim_{x \rightarrow 8^-} (-1) = -1 .$$

* Written by D. Zeilberger. Adapted by F. Knop.

Answer to 1e) -1

2. Find the following limits

$$a) \lim_{x \rightarrow 0^+} \sqrt{x} \ln x \quad b) \lim_{x \rightarrow 0^+} x^{\sin x} \quad c) \lim_{x \rightarrow \infty} \frac{(\ln x)^2}{x} .$$

Solution to 2 a) We use l'Hôpital:

$$\begin{aligned} \lim_{x \rightarrow 0^+} \sqrt{x} \ln x &= \lim_{x \rightarrow 0^+} \frac{\ln x}{x^{-1/2}} = \lim_{x \rightarrow 0^+} \frac{(\ln x)'}{(x^{-1/2})'} = \lim_{x \rightarrow 0^+} \frac{1/x}{(-1/2)x^{-3/2}} = \\ & \lim_{x \rightarrow 0^+} \frac{x^{1/2}}{(-1/2)} = \lim_{x \rightarrow 0^+} -2\sqrt{x} = \lim_{x \rightarrow 0^+} -2\sqrt{0} = 0 . \end{aligned}$$

Answer to 2a) 0

b) We must first take the \ln , getting $\ln(x^{\sin x}) = \sin x \ln x$, and writing it as a quotient we get $\frac{\ln x}{\csc x}$. Taking the limit of this we have:

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{\ln x}{\csc x} &= \lim_{x \rightarrow 0^+} \frac{(\ln x)'}{(\csc x)'} = \lim_{x \rightarrow 0^+} \frac{1/x}{-\cot x \csc x} = \\ \lim_{x \rightarrow 0^+} \frac{\tan x \sin x}{x} &= \lim_{x \rightarrow 0^+} \frac{\sin x \sin x}{(\cos x)x} = \lim_{x \rightarrow 0^+} \frac{x^2}{x} = \lim_{x \rightarrow 0^+} x = 0 . \end{aligned}$$

Now the limit of the original is the exponential of this

Answer to 2b) $e^0 = 1$

Shortcut: When x is near 0 , $\sin x$ can be replaced by x , so we get the simpler limit $\lim_{x \rightarrow 0^+} x^x$ whose \ln is $x \ln x$, and the limit

$$\begin{aligned} \lim_{x \rightarrow 0^+} x \ln x &= \lim_{x \rightarrow 0^+} \frac{\ln x}{x^{-1}} = \lim_{x \rightarrow 0^+} \frac{(\ln x)'}{(x^{-1})'} = \\ \lim_{x \rightarrow 0^+} \frac{1/x}{-x^{-2}} &= \lim_{x \rightarrow 0^+} (-x) = 0 . \end{aligned}$$

and we get quicker **Answer** $e^0 = 1$.

c) By l'Hôpital again:

$$\lim_{x \rightarrow \infty} \frac{(\ln x)^2}{x} = \lim_{x \rightarrow \infty} \frac{((\ln x)^2)'}{x'} = \lim_{x \rightarrow \infty} \frac{2(\ln x)(1/x)}{1} = \lim_{x \rightarrow \infty} \frac{2 \ln x}{x} .$$

Using l'Hôpital once again, we have that this equals

$$2 \lim_{x \rightarrow \infty} \frac{(\ln x)'}{x'} = 2 \lim_{x \rightarrow \infty} \frac{1/x}{1} = 2 \lim_{x \rightarrow \infty} \frac{1}{x} = 0 .$$

Answer to 2c) 0

3. For what values of B is the following function continuous at $x = 1$?

$$f(x) = \begin{cases} 3x^3 - x^2 - Bx, & \text{if } x > 1; \\ Bx - 2, & \text{if } x \leq 1. \end{cases}$$

Solution to 3. The limit from the left (at $x = 1$), $\lim_{x \rightarrow 1^-} f(x)$ is $3 \cdot (1)^3 - 1^2 - B \cdot 1 = 2 - B$, whereas the limit from the right $\lim_{x \rightarrow 1^+} f(x)$ is $B \cdot 1 - 2 = B - 2$. In order for $f(x)$ to be continuous at $x = 1$ we set these equal and solve. $2 - B = B - 2$ getting $B = 2$.

Answer to 3. $B = 2$

4. Find the equation of the tangent line to the curve defined by the equation $\ln(xy) + 2x - y + 1 = 0$ at the point $(\frac{1}{2}, 2)$.

Solution to 4. First simplify to get

$$\ln x + \ln y + 2x - y + 1 = 0 \quad .$$

By implicit differentiation

$$\frac{1}{x} + \frac{y'}{y} + 2 - y' = 0 \quad .$$

Now plug-in $x = 1/2$ and $y = 2$, to get

$$\frac{1}{1/2} + \frac{y'}{2} + 2 - y' = 0 \quad ,$$

getting $4 = y'/2$, and so the slope y' (at the designated point) is 8. The equation of the tangent line at the designated point $(\frac{1}{2}, 2)$ is $(y - 2) = 8(x - \frac{1}{2})$, which is $y = 8x - 2$.

Answer to 4. $y = 8x - 2$

5. Find y' in each case. Do not simplify your answer.

$$\begin{aligned} \text{a)} \quad y &= x^7 - 3x + 6 - \frac{1}{x^4} & \text{b)} \quad y &= x^{5x} \sin(x^2) & \text{c)} \quad y &= \frac{2 \tan x}{\sqrt{1-x^2}} \\ \text{d)} \quad x^4 y + 5y^6 x^3 &= 8 & \text{e)} \quad x e^{xy+3y} &= y & \text{f)} \quad f(x) &= (1+2x)^{1/x} \quad . \end{aligned}$$

Solution to 5.

a)

$$y' = (x^7 - 3x + 6 - \frac{1}{x^4})' = (x^7 - 3x + 6 - x^{-4})' = 7x^6 - 3 - (-4)x^{-5} = 7x^6 - 3 + \frac{4}{x^5} \quad .$$

Answer to 5a) $7x^6 - 3 + \frac{4}{x^5}$

b)

$$\begin{aligned}y' &= (x^{5x} \sin(x^2))' = (x^{5x})' \sin(x^2) + x^{5x} (\sin(x^2))' = (e^{5x \ln x})' \sin(x^2) + x^{5x} (\sin(x^2))' = \\ &= (5x \ln x)' (e^{5x \ln x}) \sin(x^2) + x^{5x} (2x) \cos(x^2) = \\ &= (5 \ln x + 5)(e^{5x \ln x}) \sin(x^2) + x^{5x} (2x) \cos(x^2) = \\ &= (5 \ln x + 5)x^{5x} \sin(x^2) + x^{5x} (2x) \cos(x^2) \quad .\end{aligned}$$

Answer to 5b) $(5 \ln x + 5)x^{5x} \sin(x^2) + x^{5x} (2x) \cos(x^2)$

c)

$$\begin{aligned}y' &= \left(\frac{2 \tan x}{\sqrt{1-x^2}} \right)' = \frac{(2 \tan x)' \sqrt{1-x^2} - (2 \tan x) ((1-x^2)^{1/2})'}{(\sqrt{1-x^2})^2} = \\ &= \frac{2\sqrt{1-x^2} \sec^2 x - (2 \tan x)(1/2)(1-x^2)^{-1/2} \cdot (-2x)}{1-x^2} = \\ &= \frac{2\sqrt{1-x^2} \sec^2 x + 2x(1-x^2)^{-1/2} \tan x}{1-x^2} = \frac{2 \sec^2 x}{\sqrt{1-x^2}} + \frac{2x \tan x}{\sqrt{1-x^2}^3}\end{aligned}$$

Answer to 5c) $\frac{2 \sec^2 x}{\sqrt{1-x^2}} + \frac{2x \tan x}{\sqrt{1-x^2}^3}$

d)

$$(x^4 y + 5y^6 x^3)' = 0 \quad ,$$

so

$$\begin{aligned}(x^4)'y + x^4 y' + 5y^6 (x^3)' + 5(y^6)'x^3 &= 0 \quad , \\ 4x^3 y + x^4 y' + 5y^6 (3x^2) + 5(6y^5 y')x^3 &= 0 \quad , \\ 4x^3 y + x^4 y' + 15y^6 x^2 + 30y^5 y' x^3 &= 0 \quad .\end{aligned}$$

Moving all the terms not involving y' to the right gives

$$\begin{aligned}x^4 y' + 30x^3 y^5 y' &= -4x^3 y - 15y^6 x^2 \quad . \\ (x^4 + 30x^3 y^5) y' &= -4x^3 y - 15y^6 x^2 \quad .\end{aligned}$$

Hence

$$y' = \frac{-4x^3 y - 15y^6 x^2}{x^4 + 30x^3 y^5}$$

Since we are not supposed to simplify, we have

Answer to 5d) $y' = \frac{-4x^3 y - 15y^6 x^2}{x^4 + 30x^3 y^5}$

e)

$$(x e^{xy+3y})' = y' \quad ,$$

$$\begin{aligned}
x' e^{xy+3y} + x(e^{xy+3y})' &= y' \quad , \\
e^{xy+3y} + x e^{xy+3y} \cdot (xy + 3y)' &= y' \quad , \\
e^{xy+3y} + x e^{xy+3y} \cdot (x'y + xy' + 3y') &= y' \quad , \\
e^{xy+3y} + x e^{xy+3y} \cdot (y + (x + 3)y') &= y' \quad , \\
e^{xy+3y} + xy e^{xy+3y} + (x + 3)xy' e^{xy+3y} &= y' \\
(1 + xy)e^{xy+3y} &= y'(1 - x(x + 3)e^{xy+3y})
\end{aligned}$$

Hence

$$y' = \frac{e^{xy+3y}(1 + xy)}{1 - x(x + 3)e^{xy+3y}} .$$

Answer to 5e) $y' = \frac{e^{xy+3y}(1+xy)}{1-x(x+3)e^{xy+3y}}$

f)

$$\begin{aligned}
(\ln f(x))' &= \left(\frac{\ln(1 + 2x)}{x}\right)' \\
\frac{f'(x)}{f(x)} &= \frac{(\ln(1 + 2x))'x - \ln(1 + 2x) \cdot x'}{x^2} = \frac{(2x/(1 + 2x)) - \ln(1 + 2x)}{x^2}
\end{aligned}$$

Hence

$$f'(x) = f(x) \frac{(2x/(1 + 2x)) - \ln(1 + 2x)}{x^2} = (1 + 2x)^{1/x} \frac{2x/(1 + 2x) - \ln(1 + 2x)}{x^2} .$$

Answer to 5f) $(1 + 2x)^{1/x} \frac{2x/(1+2x) - \ln(1+2x)}{x^2}$

6. State the formal definition of the derivative of the function $f(x)$. Use the definition to calculate $f'(x)$ for $f(x) = \sqrt{3 - 5x}$.

Solution to 6.

The formal definition of the derivative of a function $f(x)$ is

$$f'(x) = \lim_{s \rightarrow x} \frac{f(s) - f(x)}{s - x} .$$

Note: There is another version with h and $\lim_{h \rightarrow 0}$, but for *this* problem it is more convenient to use this one.

$$\begin{aligned}
f'(x) &= \lim_{s \rightarrow x} \frac{\sqrt{3 - 5s} - \sqrt{3 - 5x}}{s - x} = \lim_{s \rightarrow x} \frac{(\sqrt{3 - 5s} - \sqrt{3 - 5x})(\sqrt{3 - 5s} + \sqrt{3 - 5x})}{(s - x)(\sqrt{3 - 5s} + \sqrt{3 - 5x})} \\
&= \lim_{s \rightarrow x} \frac{(\sqrt{3 - 5s})^2 - (\sqrt{3 - 5x})^2}{(s - x)(\sqrt{3 - 5s} + \sqrt{3 - 5x})} = \lim_{s \rightarrow x} \frac{(3 - 5s) - (3 - 5x)}{(s - x)(\sqrt{3 - 5s} + \sqrt{3 - 5x})}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{s \rightarrow x} \frac{-5(s-x)}{(s-x)(\sqrt{3-5s} + \sqrt{3-5x})} = \\
\lim_{s \rightarrow x} \frac{-5}{\sqrt{3-5s} + \sqrt{3-5x}} &= \frac{-5}{\sqrt{3-5x} + \sqrt{3-5x}} = \frac{-5}{2\sqrt{3-5x}} .
\end{aligned}$$

Answer to 6. $\frac{-5}{2\sqrt{3-5x}}$ (but it had to be derived from the definition!).

7. Suppose that $S(x) = \sqrt{x}$ for $x \geq 0$, and let f and g be differentiable functions about which the following is known

$$f(3) = 2, \quad f'(3) = 7, \quad g(3) = 4, \quad g'(3) = 5 .$$

Compute the following

$$(f+g)'(3) \quad (f \cdot g)'(3), \quad \left(\frac{f}{g}\right)'(3), \quad (S \circ g)'(3), \quad \left(\frac{f \cdot g}{f-g}\right)'(3) .$$

Solution to 7.

$$(f+g)'(3) = f'(3) + g'(3) = 7 + 5 = 12 .$$

$$(f \cdot g)'(3) = f'(3) \cdot g(3) + f(3) \cdot g'(3) = 7 \cdot 4 + 2 \cdot 5 = 38 .$$

$$\left(\frac{f}{g}\right)'(3) = \frac{f'(3)g(3) - f(3)g'(3)}{g(3)^2} = \frac{7 \cdot 4 - 2 \cdot 5}{4^2} = \frac{18}{16} = \frac{9}{8} .$$

$$(S \circ g)'(3) = S'(g(3))g'(3) = S'(4) \cdot 5 = 5 \cdot (1/2)(4)^{-1/2} = 5/4 .$$

$$\begin{aligned}
&\left(\frac{f \cdot g}{f-g}\right)'(3) = \\
&\left(\frac{(f \cdot g)'(f-g) - (f \cdot g)(f-g)'}{(f-g)^2}\right)(3) = \\
&\left(\frac{(f'g + fg')(f-g) - (f \cdot g)(f' - g')}{(f-g)^2}\right)(3) = \\
&\left(\frac{f^2g' - f'g^2}{(f-g)^2}\right)(3) = \\
&\frac{f(3)^2g'(3) - f'(3)g(3)^2}{(f(3) - g(3))^2} = \\
&\frac{2^2 \cdot 5 - 7 \cdot 4^2}{(2 - 4)^2} = -23 .
\end{aligned}$$

Answer to 7. 12, 38, 9/8, 5/4, -23

8. A ladder of length 15 feet is leaning against a wall when its base begins to slide along the floor, away from the wall. By the time the base is 12 feet away from the wall, the base is moving at the rate of 5 ft/sec. How fast is the top of the ladder sliding down the wall at that moment? How fast is the area of the triangle formed by the ladder, the wall, and the floor changing at that time?

Solution to 8. This is a related rates problem. Let's call the height of the top of the ladder y , and the distance of the foot of the ladder from the wall x . By Pythagoras,

$$x^2 + y^2 = 15^2 \quad .$$

Taking derivatives with respect to time (using the chain rule in its abstract setting, a.k.a. implicit differentiation), we get

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \quad .$$

Right now $x = 12$ and $dx/dt = 5$. We also need y right now. Using the relation, $x^2 + y^2 = 15^2$ we plug-in $x = 12$ and solve for y : $12^2 + y^2 = 15^2$, getting $y = 9$. Substituting these values into the derived relation we get

$$2 \cdot 12 \cdot 5 + 2 \cdot 9 \frac{dy}{dt} = 0 \quad ,$$

and solving for dy/dt , we get $dy/dt = -20/3$. Since the sign is negative, it is sliding **down** (which is also obvious from common sense).

Answer to the first part of 8. The top of the ladder is sliding down at a speed of $20/3$ ft/sec.

The **area**, let's call it A , is given by the relation $A = xy/2$. Taking derivative with respect to time (using the product rule), we get

$$\frac{dA}{dt} = \frac{1}{2} \left(x \frac{dy}{dt} + y \frac{dx}{dt} \right) \quad ,$$

and we plug-in the above numbers, getting

$$\frac{dA}{dt} = \frac{1}{2} (12 \cdot (-20/3) + 9 \cdot 5) = -35/2 \quad ,$$

Answer to second part of 8. The area is decreasing at a rate of $35/2$ square feet per second.

9. A farmer with 450 feet of fencing wants to enclose the four sides of a rectangular region into four pens of equal size with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?

Solution to 9. Let's call the length of the side parallel to the three internal walls x , and the length of the other side (perpendicular to these walls) y .

We have to **maximize** the **area**. So the **Goal** function is xy . We also have the **constraint**: $2y + 5x = 450$. Solving for y , we get $y = (450 - 5x)/2$, and plugging this into the goal function, we express it in terms of x alone, getting $f(x) = x(450 - 5x)/2 = (450/2)x - (5/2)x^2$. To find the maximum, we take the derivative $f'(x) = 450/2 - 5x$, and then set it equal to 0. Solving $450/2 - 5x = 0$, we get $x = 45$. Plugging this into $f(x)$ we get that the maximal total area is $f(45) = 45 \cdot (450 - 5 \cdot 45)/2 = 45^2 \cdot 5/2 = 10125/2$.

Ans to 9. 10125/2.

Note: In some problems, they might ask you about the **dimensions** of the optimal rectangle. Then the answer would be $45 \times (225/2)$.

10. Let $f(x) = \frac{3x}{x^2-1}$. Find the domain of the function, the intervals where $f(x)$ is increasing or decreasing, maximum and minimum, the concavity and inflection points, horizontal and vertical asymptotes of the graph. Then sketch the graph of $f(x)$.

Solution to 10. Domain: $\{x|x \neq -1, 1\}$. Even though the horizontal, and vertical asymptotes are mentioned **last** in the statement of the problem, it is best to start with them, since they contain the most important information.

horizontal asymptotes:

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{3x}{x^2-1} = \lim_{x \rightarrow \infty} \frac{3x}{x^2} = 0 \quad .$$

So $y = 0$ (alias the x-axis) is a horizontal asymptote.

vertical asymptotes: Set the denominator equal to 0, we get $x = -1$ and $x = 1$.

Let's consider each vertical asymptote on its own. Plugging-in a value of x slightly less than -1 , say -1.01 , we get $-BIG$, so this means that $\lim_{x \rightarrow -1^-} f(x) = -\infty$, and the curve would drop down to "hell" ($-\infty$) as it approaches the vertical asymptote $x = -1$. Plugging-in a value of x slightly bigger than -1 , say $-.99$ we get that $f(-.99) = BIG$, so the curve "comes down from heavens" right after $x = -1$. Similarly the curve goes down to hell as x approaches $x = 1$, and emerges from heavens right after $x = 1$.

It is also a good idea to find the x -intercept (solve $y = 0$, getting $(0,0)$) and the y -intercept, plugging-in $x = 0$, getting, once again $(0,0)$. So the curve intersects the x -axis only at the origin $(0,0)$.

Notice that we haven't taken derivatives yet, but we can already pretty much draw the function. When x is very negative, it is almost coincident with the x -axis (the horizontal Asymptote) but it is below the x -axis, since $f(x)$ is negative there, and then it is dropping down-down right before $x = -1$. By inspection you can see that the curve is **concave down** at the interval $-\infty < x \leq -1$

Then from $x = -1$ to $x = 1$ it is a steady decrease (since at $x = -1$ it is coming from $y = \infty$ dropping to $y = -\infty$ at $x = 1$. Somewhere between these, there should be an inflection point, that by symmetry is $(0,0)$. Also for $x > 1$ it dropping from ∞ to 0. So you can draw the curve without any calculus! Also it is clear that there are no local maxima and minima.

But now we can do some calculus.

By the quotient rule

$$f'(x) = -3 \frac{x^2 + 1}{(x^2 - 1)^2} \quad ,$$

this is always negative, so **intervals of increase:** none.

intervals of decrease: $(-\infty, -1), (-1, 1), (1, \infty)$.

To find potential (local) max or min. points, solve $f'(x) = 0$, which has no roots. Hence

max and min. points: none.

Using the quotient rule one more time, we get

$$f''(x) = \frac{6x(x^2 + 3)}{(x^2 - 1)^3} \quad .$$

The inflection point(s) are where $f''(x) = 0$. Setting the numerator equal to 0, we get $x = 0$. The y coordinate is $f(0) = 0$. So

inflection point: $(0, 0)$.

The transition points in the life of this function are $x = -1, x = 0$, and $x = 1$. Plugging-in random values in each interval we get

The interval $(-\infty, -1)$ is **concave down** (since. e.g. $f''(2) < 0$).

The interval $(-1, 0)$ is **concave up** (since. e.g. $f''(-\frac{1}{2}) > 0$).

The interval $(0, 1)$ is **concave down** (since. e.g. $f''(\frac{1}{2}) < 0$).

The interval $(1, \infty)$ is **concave up** (since. e.g. $f''(2) > 0$).

Answer to 10. max. and min. points: none; intervals of increase: none; intervals of decrease: $(-\infty, -1), (-1, 1), (1, \infty)$; inflection point: $(0, 0)$; intervals of concave down: $(-\infty, -1), (0, 1)$; intervals of concave up: $(-1, 0), (1, \infty)$; horizontal asymptote: $y = 0$; vertical asymptotes: $x = -1$ and $x = 1$.

11. Sketch the graph of $f(x)$ which satisfies the following conditions $f'(1) = f'(-1) = 0$, $f'(x) < 0$ if $|x| < 1$, $f'(x) > 0$ if $1 < |x| < 2$, $f'(x) = -1$ if $|x| > 2$, $f''(x) < 0$ if $-2 < x < 0$, inflection point $(0, 1)$.

Solution to 11. First let's spell everything out w/o the annoying absolute value symbol $||$.

For $-\infty < x < -2$: $f'(x) = -1$.

For $-2 < x < -1$: $f'(x) > 0$.

For $-1 < x < 1 : f'(x) < 0$.

For $1 < x < 2 : f'(x) > 0$.

For $2 < x < \infty : f'(x) = -1$.

Now we are almost ready to sketch, noting that at $x = -2$ and $x = 1$ we have local minima, since the derivative changes sign from negative to positive, and at $x = -1$ and $x = 2$ we have local maxima, since $f'(x)$ changes sign from positive to negative. We also have an inflection point at $(0, 1)$ and since $f''(x) < 0$ between $x = -2$ and $x = 0$, it is concave down there, and hence between $x = 0$ and $x = 2$ it is concave up.

To actually draw a curve fitting all these specifications, draw a straight line with slope -1 coming from $-\infty$ to the point $(-2, 0)$, then draw a hat-like curve that peaks at $x = -1$ and inflects at the point $(0, 1)$ turning into a cup-like arc that bottoms at $x = 1$ and goes up to $x = 2$ and then it suddenly becomes a straight line with slope -1 .

12. Find the linearization of $f(x) = \sqrt{x+1}$ at $a = 15$ and use it to find an approximation to $\sqrt{15}$ and $\sqrt{17}$. Use the second derivative to determine whether the estimate is greater or less than the actual value.

Solution to 12. The linearization of a function $f(x)$ at $x = a$ is $L(x) = f(a) + f'(a)(x - a)$. In this problem $f(x) = (x + 1)^{1/2}$, so

$$f'(x) = (1/2)(x + 1)^{-1/2} = \frac{1}{2\sqrt{x+1}} \quad .$$

We have $f(a) = f(15) = \sqrt{1+15} = \sqrt{16} = 4$, and $f'(a) = f'(15) = \frac{1}{2\sqrt{15+1}} = 1/8$. Hence

Answer to first part of 12. $L(x) = 4 + \frac{1}{8}(x - 15)$

Now $\sqrt{15} = f(14)$, and plugging $x = 14$, we get $L(14) = 4 + \frac{1}{8}(14 - 15) = \frac{31}{8}$, hence

Answer to second part of 12. $\sqrt{15}$ is approximately $\frac{31}{8}$

Also $\sqrt{17} = f(16)$, and plugging $x = 16$, we get $L(16) = 4 + \frac{1}{8}(16 - 15) = \frac{33}{8}$, hence

Answer to third part of 12. $\sqrt{17}$ is approximately $\frac{33}{8}$

$$f''(x) = (-1/4)(x + 1)^{-3/2} = \frac{-1}{4\sqrt{x+1}^3} \quad .$$

Since $f''(15) = -1/256$ is **negative**, the above approximations to $\sqrt{15}$ and $\sqrt{17}$ are **larger** than the actual value.

Answer to fourth part of 12. Both are larger than the actual values.

13. Find the absolute maximum and the absolute minimum values of $f(x) = \frac{x}{x^2+1}$ in $[0, 2]$.

Solution to 13. Using the quotient rule, we have

$$f'(x) = \frac{x'(x^2 + 1) - x(x^2 + 1)'}{(x^2 + 1)^2} = \frac{1 \cdot (x^2 + 1) - x \cdot 2x}{(x^2 + 1)^2} = \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2} .$$

We have to solve $f'(x) = 0$. Setting the **numerator** equal to 0, we get $1 - x^2 = 0$, so $(1 - x)(1 + x) = 0$ and we get the two roots $x = -1$ and $x = 1$. But $x = -1$ does not satisfy the **residency requirement**, i.e. it does not belong to the interval $[0, 2]$, so we have the three **finalists**: 0, 1, 2. (The endpoints of the interval always participate in the **final contest**).

Now let's plug these numbers in $f(x) = \frac{x}{x^2 + 1}$:

$$f(0) = \frac{0}{0^2 + 1} = 0 \quad f(1) = \frac{1}{1^2 + 1} = \frac{1}{2} = .5 \quad f(2) = \frac{2}{2^2 + 1} = \frac{2}{5} = .4 .$$

Answer to 13. Absolute minimum value is 0, Absolute maximum value is 1/2.

14. Let $f(x) = 3x^3 - 2x^2 + x - 1$. Show that $f(x)$ must have a real root in $[0, 1]$.

Solution to 14. The given function $f(x)$ is continuous (because it is a polynomial), Plug-in the leftmost end, 0, into $f(x)$, getting $f(0) = -1$. Plug-in the rightmost end, 1, into $f(x)$, getting $f(1) = 3 \cdot 1^3 - 2 \cdot 1^2 + 1 - 1 = 1$. Since these two numbers -1 and 1 have **opposite signs** (one is positive and the other is negative), it follows by the **intermediate value theorem**, that there must be a $c \in (0, 1)$ such that $f(c) = 0$. In other words, $f(x) = 0$ must have a real root in $[0, 1]$.

15. Suppose that f is continuous on $[0, 4]$, $f(0) = 1$ and $2 \leq f'(x) \leq 5$ for all $x \in (0, 4)$. Show that $9 \leq f(4) \leq 21$.

Solution to 15. We must use the **Mean Value Theorem**. If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b) then there exists a c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} .$$

In this problem $a = 0$ and $b = 4$ so, there exists a c in $(0, 4)$ such that

$$f'(c) = \frac{f(4) - f(0)}{4 - 0} = \frac{f(4) - 1}{4 - 0}$$

But we know that $2 \leq f'(x) \leq 5$ for *every* $x \in (0, 4)$, in particular for the c , whatever it is. Hence

$$2 \leq \frac{f(4) - 1}{4} \leq 5 .$$

Multiplying by 4, we get (since 4 is positive, the inequalities are preserved) $8 \leq f(4) - 1 \leq 20$, and adding 1 everywhere, we get $9 \leq f(4) \leq 21$.

16. Use two steps of Newton's method to approximate the root of the equation $x^4 + x - 4 = 0$ in the interval $[1, 2]$.

Solution to 16, Recall that **Newton's method** is:

$$x_n = x_o - \frac{f(x_o)}{f'(x_o)} \quad ,$$

where x_o is the **old** approximation and x_n is the **new** approximation. Implementing this here, we get $f(x) = x^4 + x - 4$, and $f'(x) = 4x^3 + 1$, so for this problem we have

$$x_n = x_o - \frac{x_o^4 + x_o - 4}{4x_o^3 + 1} \quad .$$

$f(1) = -2$ and $f(2) = 14$. Since -2 is so much closer to 0 than 14, it is reasonable to take as the **initial approximation** $x_o = 1$. We get

$$x_n = 1 - \frac{1^4 + 1 - 4}{4 \cdot 1^3 + 1} = 1 - \frac{-2}{5} = 7/5 \quad .$$

Now let $x_0 = 7/5$ and we get

$$x_n = 7/5 - \frac{(7/5)^4 + 7/5 - 4}{4 \cdot (7/5)^3 + 1} \quad .$$

Since you don't have a calculator, you leave it like this. (It turns out to be 9703/7485).

Answer to 16. $7/5 - \frac{(7/5)^4 + 7/5 - 4}{4 \cdot (7/5)^3 + 1} = 9703/7485$

17. Find the most general antiderivative of the functions

$$a) \quad f(x) = 1 - x^3 + 5x^5 - 3x^7 \quad b) \quad g(x) = \frac{5 - 4x^3 + 3x^6}{x^6}$$

$$c) \quad h(x) = 3e^x + 7 \sec^2 x + 5(1 - x^2)^{-\frac{1}{2}}.$$

Solution to 17. Recall the formula

$$\int x^n = \frac{x^{n+1}}{n+1} + C \quad .$$

a)

$$\int f(x)dx = \int (1 - x^3 + 5x^5 - 3x^7)dx = x - \frac{x^4}{4} + 5\frac{x^6}{6} - 3\frac{x^8}{8} = x - (1/4)x^4 + (5/6)x^6 - (3/8)x^8 + C \quad .$$

Answer to 17a) $x - (1/4)x^4 + (5/6)x^6 - (3/8)x^8 + C$

b) First **break it up**, then integrate **term by term**.

$$\int g(x)dx = \int \frac{5 - 4x^3 + 3x^6}{x^6}dx = \int (5 \cdot x^{-6} - 4x^{-3} + 3)dx = 5x^{-5}/(-5) - 4(x^{-2}/(-2)) + 3x =$$

$$-x^{-5} + 2x^{-2} + 3x = \frac{-1}{x^5} + \frac{2}{x^2} + 3x + C \quad .$$

Answer to 17b) $\frac{-1}{x^5} + \frac{2}{x^2} + 3x + C$

c) Recall (or look up in the formula sheet) that $(\tan x)' = \sec^2 x$, hence the **antiderivative** of $\sec^2 x$ is $\tan x + C$. Also $(\sin^{-1} x)' = (1 - x^2)^{-\frac{1}{2}}$, hence the antiderivative of $(1 - x^2)^{-\frac{1}{2}}$ is $\sin^{-1} x + C$. Of course the antiderivative of e^x is $e^x + C$. So

$$\int f(x)dx = \int (3e^x + 7\sec^2 x + 5(1 - x^2)^{-\frac{1}{2}})dx = 3e^x + 7\tan x + 5\sin^{-1} x + C \quad .$$

Answer to 17c) $3e^x + 7\tan x + 5\sin^{-1} x + C$

18. Evaluate the integral if it exists.

$$\begin{aligned} \text{a)} \quad & \int_1^9 \frac{\sqrt{u} - 2u^2}{u} du. \quad \text{b)} \quad \int_0^2 y^2 \sqrt{1 + y^3} dy. \quad \text{c)} \quad \int_1^5 \frac{dt}{(t-4)^2} \quad . \\ \text{d)} \quad & \int_0^1 t^2 \cos(t^3) dt. \quad \text{e)} \quad \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx. \end{aligned}$$

Solution to 18.

a)

$$\begin{aligned} \int_1^9 \frac{\sqrt{u} - 2u^2}{u} du &= \int_1^9 (u^{-\frac{1}{2}} - 2u) du = u^{\frac{1}{2}} / (\frac{1}{2}) - \frac{2u^2}{2} \Big|_1^9 = 2\sqrt{u} - u^2 \Big|_1^9 = \\ & [2\sqrt{9} - 9^2] - [2\sqrt{1} - 1^2] = -76 \quad . \end{aligned}$$

Answer to 18a) -76

b) We make the substitution $u = 1 + y^3$ which entails $du/dy = 3y^2$ hence $dy = du/(3y^2)$. Also the limits of the new integral must be computed. When $y = 0$, $u = 1$, and when $y = 2$, $u = 9$. So

$$\begin{aligned} \text{b)} \quad \int_0^2 y^2 \sqrt{1 + y^3} dy &= \int_1^9 y^2 \sqrt{u} \frac{du}{3y^2} = \frac{1}{3} \int_1^9 \sqrt{u} du = \frac{1}{3} \int_1^9 u^{\frac{1}{2}} du = \frac{1}{3} \frac{1}{\frac{3}{2}} u^{\frac{3}{2}} \Big|_1^9 = \\ & \frac{2}{9} \sqrt{u^3} \Big|_1^9 = \frac{2}{9} (\sqrt{9^3} - \sqrt{1^3}) = \frac{2}{9} (3^3 - 1^3) = \frac{52}{9} \quad . \end{aligned}$$

Answer to 18b) $52/9$

c) Here the substitution is $u = t^3$ giving $du/dt = 3t^2$, and so $dt = du/(3t^2)$. As for the limits of integration, when $t = 0$ $u = 0$ and when $t = 1$, $u = 1$.

$$\int_0^1 t^2 \cos(t^3) dt = \int_0^1 t^2 \cos(u) \frac{du}{3t^2} = \frac{1}{3} \int_0^1 \cos(u) du = \frac{1}{3} \sin u \Big|_0^1 = \frac{1}{3} (\sin 1 - \sin 0) = (\sin 1)/3 \quad .$$

Answer to 18c) $(\sin 1)/3$

d) This is an *indefinite integral* since there are no limits of integration. The answer should be an **expression in x** . In the method of **substitution**, we must translate back at the end.

Take $u = \sqrt{x} = x^{\frac{1}{2}}$, so $du/dx = (1/2)x^{-\frac{1}{2}}$ and $dx = 2x^{\frac{1}{2}}du = 2\sqrt{x}du$. We have

$$e) \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \int \frac{e^u}{\sqrt{x}} 2\sqrt{x} du = 2 \int e^u du = 2e^u = 2e^{\sqrt{x}} + C \quad .$$

Answer to 18d) $2e^{\sqrt{x}} + C$

19. Suppose $f''(x) = -3x + \cos(\pi x)$ and $f(1) = 2$ and $f'(1) = -1$. What is $f(5)$?

Solution to 19. The general antiderivative of $-3x + \cos(\pi x)$ is $f'(x) = -3x^2/2 + \sin(\pi x)/\pi + A$, where A is a general constant. To find A , we plug-in $x = 1$, and get $f'(1) = -3 \cdot 1^2/2 + \sin(\pi)/\pi + A = -3/2 + 0 + A$. On the other hand, in this specific problem, we are told that $f'(1) = -1$, so we get the equation $-1 = A - 3/2$, and solving for A , we get $A = 1/2$. So

$$f'(x) = -(3/2)x^2 + \sin(\pi x)/\pi + 1/2 \quad .$$

To get $f(x)$, we take the antiderivative of the above, getting $f(x) = -(1/2)x^3 - \cos(\pi x)/\pi^2 + x/2 + B$, where B is a constant yet to be determined. Plugging-in $x = 1$ into this yields $f(1) = -1/2 - \cos(\pi)/\pi^2 + 1/2 + B = 1/\pi^2 + B$. On the other hand, in this specific problem, we are told that $f(1) = 2$, so we get the equation $2 = 1/\pi^2 + B$, so $B = 2 - 1/\pi^2$. Hence

$$f(x) = -(1/2)x^3 - \cos(\pi x)/\pi^2 + x/2 + 2 - 1/\pi^2$$

Finally, plugging-in $x = 5$, we get $f(5) = -(1/2)5^3 - \cos(5\pi)/\pi^2 + 5/2 + 2 - 1/\pi^2 = -125/2 + 1/\pi^2 + 5/2 + 2 - 1/\pi^2 = -120/2 + 2 = -60 + 2 = -58$.

Answer to 19. -58

20. Find the derivative of the following functions:

$$a) \quad F(x) = \int_{\pi}^x \tan(s^2) ds \quad b) \quad g(x) = \int_1^{\cos x} \sqrt{1-t^2} dt \quad .$$

Solution to 20.a) Since the upper-limit of integration is **identical** to the argument of F , (both are x), the derivative is simply the integrand with s replaced by x .

Answer to 20a) $F'(x) = \tan(x^2)$

Solution to 20b) Now the upper-limit of integration, $\cos x$ is **not** identical with the argument of g , which is x . We must first use the chain rule, multiplying top and bottom by $d \cos x$ (so to speak!).

$$g'(x) = \frac{d}{dx} \int_1^{\cos x} \sqrt{1-t^2} dt = \frac{d \cos x}{dx} \cdot \frac{d}{\cos x} \int_1^{\cos x} \sqrt{1-t^2} dt \quad .$$

The first piece is a straightforward differentiation, while the second is done exactly as in a): replace t by $\cos x$ in the integrand. Hence $F'(x) = -\sin x \sqrt{1 - \cos^2 x}$. Using the famous trig. identity this is $-\sin^2 x$.

Answer to 20b) $g'(x) = -\sin^2 x$

21. Find the area bounded by the two curves (a) $y_1 = 2x^2$ and $y_2 = 8x$. (b) $y_1 = \sin x$ and $y_2 = \cos x$, $0 \leq x \leq \frac{\pi}{2}$

Solution to 21. Recall that the area is (if the two curves meet only at two points)

$$\int_a^b (\text{TOP} - \text{BOTTOM}) dx \quad ,$$

where a and b are the two roots of $y_1 = y_2$. Solving $2x^2 = 8x$ yields $2x(x - 4) = 0$ so $x = 0$ and $x = 4$, and indeed there are only two meeting points. To find who's on top just plug-in any x in the interval $(0, 4)$. $x = 1$ is a good choice. When $x = 1$, $y_1 = 2$ and $y_2 = 8$, since $2 < 8$ we know that $y_2 = 8x$ is on top and $y_1 = 2x^2$ is at the bottom. So the area is

$$\int_0^4 (8x - 2x^2) dx = 4x^2 - (2/3)x^3 \Big|_0^4 = 4 \cdot 4^2 - (2/3) \cdot 4^3 = \frac{64}{3} \quad .$$

Answer to 21a) $\frac{64}{3}$

b) The curves $y_1 = \sin x$ and $y_2 = \cos x$ meet when $\sin x = \cos x$, and that happens when $x = \pi/4$. There are two parts (equal by symmetry). The one bounded by $y = \sin x$, $y = \cos x$, and the y -axis (alias $x = 0$), and the one bounded by $y = \sin x$, $y = \cos x$, and $x = \pi/2$. The first one happens between $x = 0$ and $x = \pi/4$ and the second between $x = \pi/4$ and $x = \pi/2$. In the first half $y = \cos x$ is on top and $y = \sin x$ is at the bottom, and at the second half it is the other way. Hence the area is

$$\int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx = (\sin x + \cos x) \Big|_0^{\pi/4} + (-\cos x - \sin x) \Big|_{\pi/4}^{\pi/2} =$$

$$(\sin(\pi/4) + \cos(\pi/4)) - (\sin(0) + \cos(0)) + (-\cos(\pi/2) - \sin(\pi/2)) - (-\cos(\pi/4) - \sin(\pi/4)) = 2\sqrt{2} - 2.$$

Answer to 21b) $2\sqrt{2} - 2$